Method to determine correction factors for Section Modulus and Bending Inertia Equations of Wings

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This document and more information can be found on my website Wingbike - a Human Powered Hydrofoil.

Abstract

For the calculation of the section modulus and the bending inertia of foils, Besnard [1] and Brooks [2] have provided simple approximations by representing the foil cross-section by a rectangular shape. This involves using correction factors (for chord and thickness) that are specific for a given foil. In this paper a method is presented to calculate accurate and specific correction factors for any foil shape when the Besnard/Brooks approximations do not suffice anymore. The principle is to calculate the bending inertia using a numerical scheme (once). From there on, the correction factors can be derived.

This is illustrated for the ClarkY foil resulting in more accurate equations (solid):

α (correction factor for chord) = 0,294

β (correction factor for thickness = 1,224

These factors lead to the following equations for the ClarkY foil:

$$S = 0{,}0733CT^2$$

$$I = 0{,}0449CT^3$$

The equations for hollow foils are presented in this paper as well.

The method has been validated with an ellipse and yielded an accuracy of more than 99%. It can be used for other foils and shapes like struts and beams.

A quick reference guide with all the equations can be found in appendix 4.

Introduction

When calculating stress and deflection in wings, the section modulus and the bending inertia are required. When deflection is calculated numerically and the foil is tapered, these values have to be calculated numerous times as the chord and thickness gradually change along the span. If the corresponding section modulus and bending inertia are calculated numerically as well, each step in the wing deflection calculation involves a large amount of interrelated numerical calculations.

These calculations can be simplified by representing this cross-section of the foil by a rectangular shape (corrected for chord and thickness), although this can introduce errors.

Besnard [1] and Brooks [2] have provided general approximations for the section modulus and bending inertia that can be used (for solid foils):

<u>Besnard</u>

Section modulus

$$S = \frac{(0{,}75C)(0{,}85T)^2}{6} = 0{,}0903CT^2 \qquad (1)$$

Bending inertia

$$I = \frac{(0{,}75C)(0{,}85T)^3}{12} = 0{,}0384CT^3 \qquad (2)$$

<u>Brooks</u>

Section modulus

$$S = \frac{0,45CT^2}{6} = 0,075CT^2 \quad (3)$$

Bending inertia

$$I = \frac{0,45CT^3}{12} = 0,0375CT^3 \quad (4)$$

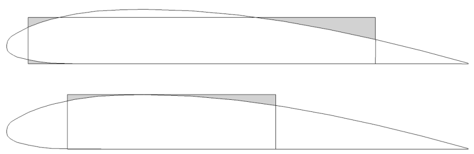
Where

C= chord [m]

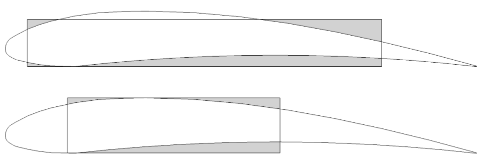
T = thickness [m]

A comparison between the two sources for a ClarkY foil can be found in [3].

The principle is to substitute the cross-section of the foil by a rectangular shape, corrected for chord and thickness. Below, the principle of the Besnard / Brooks approximation for a ClarkY foil is presented.



The difference between the two sources is that Besnard (upper) corrects the chord and thickness, whereas Brooks (lower) merely corrects the chord, yielding different factors in the equations 1 to 4 below.
It isn't hard to imagine that these approximations become less accurate for specific shapes as can be seen below.



The factors in the equations (0,0903; 0,0384; 0,075; 0,0375) are partially[1] correction factors and suitable for general purposes although they were derived by Besnard and Brooks for specific foils. If one requires more accuracy, these factors need to be re-determined for the specific foil.
A solution can be found, as will be explained in the next sections.

The bending inertia of a foil section can be calculated with a numerical scheme as well. As stated before, this is not very practical when performing numerical wing tip deflection calculations for tapered foils.
The numerical method (in Excel) is a numerical integration over the cross section of the foil. It is more accurate than the general approximations presented by Brooks and Besnard, as it accounts for the unique shape of the cross-section.
By performing a one-off numerical calculation for the specific foil cross-section, the relevant correction factors for the "simplified" equations can be derived from it.
In this way, simple and yet accurate equations can be obtained and can be used in the wing tip deflection calculations for each interval and thus simplifying the calculations.
Note that the numerical method presented in this document is for the bending inertia only, but since bending inertia and section modulus are related, the rest can be derived.

We will illustrate this method for a ClarkY foil, but the method is universal for all types and shapes of foils, even struts and beams. In Appendix 3, the method is validated for a beam with an elliptical cross-section and yields an accuracy of 99%.

1. Bending Inertia – numerical method

The basis is described by Drela [4] and it consists of three steps.
First calculate the area of the cross-section, then calculate the neutral surface. Finally, from this, the bending inertia can be calculated. The paper uses three integrals (equations 1, 2, and 3). Without going into the details of the theory, an explanation of a numerical method is given here.

[1] Part of the factor consist of 1/6 or 1/12.

The basis is to make a table in Excel (see appendix 1). In this example, we will be using the coordinates of ClarkY as it is a very basic foil with a nice flat bottom.

Column 1, 2 and 3 are the coordinates that specify the shape of the cross-section. Where x is the coordinate along the chord, and Zu and Zl represent the upper and lower curves.

The first step is to calculate the area of the cross-section. This is done by calculating the area beneath the upper curve and beneath the lower curve by means of the Trapezoidal Rule [5]. The total surface of the cross-section is obtained by the summation of column 4 subtracted by the summation of column 5.

Next step is to calculate the neutral surface (z) in column 6. Calculate for each interval:

$$\frac{1}{2}\left[Z_u^2 - Z_l^2\right]\Delta x \qquad (5)$$

After summation of column 6, divide this total by the area previously calculated. Now the neutral surface (z) is obtained.

Finally the bending inertia can be calculated. Calculate for each interval (column 7):

$$\frac{1}{3}\left[(Z_u - z)^3 - (Z_l - z)^3\right]\Delta x \qquad (6)$$

The sum of column 7 yields the bending inertia.

For a solid ClarkY foil with a chord of 150 mm and a thickness of 20 mm, this leads to a bending inertia of 53.821 mm^4 (see appendix 1).

With the figures obtained in this scheme, the section modulus can be calculated: dividing the bending inertia by y = 4.398 mm^3.

2. Matching the equations

The bending inertia equation with general correction factors for any foil is given by :

$$I = \frac{(\alpha C)(\beta T)^3}{12} \qquad (7)$$

Where

α = correction factor chord	[-]
β = correction factor thickness	[-]
C = chord	[m]
T= thickness	[m]

Similar the section modulus equation is given by:

$$S = \frac{(\alpha C)(\beta T)^2}{6} \qquad (8)$$

Where the factors α and β are determined as (see appendix 2 on how these factors were derived):

$$\alpha = \frac{3S^3}{2CI^2} \qquad (9)$$

And

$$\beta = \frac{2I}{ST} \qquad (10)$$

With the output of the numerical scheme presented in appendix 1:

S =	4.398	mm^3
I =	53.821	mm^4
C =	150	mm
T =	20	mm

One can calculate the correction factors:

α = 0,294

β = 1,224

Resulting in a section modulus equation (ClarkY):

$$S = \frac{(0{,}294C)(1{,}224T)^2}{6} = 0{,}0733CT^2 \qquad (11)$$

And bending inertia equation (ClarkY):

$$I = \frac{(0{,}294C)(1{,}224T)^3}{12} = 0{,}0449CT^3 \qquad (12)$$

3. Determine the factors for hollow foils

The factors are identical for hollow foils, only the equations differ. Deriving the general equations for hollow foils has been explained in [3], therefore we will only present the results:

For Section Modulus:

$$S = 0{,}0733C\left\{T^2 - \frac{(T-2Sk)^3}{T}\right\} \qquad (11)$$

Where

C = outer chord	[m]
T = outer thickness	[m]
S_k = Skin Thickness	[m]

And Bending Inertia:

$$I = 0{,}0449C\left\{T^3 - (T-2Sk)^3\right\} \qquad (12)$$

Important Notes to equations 11 & 12:

Firstly, these two equations are only valid if the thickness of the skin is << chord. For the exact solutions, see [3].

Secondly, the equations assume a constant skin thickness along the span of a tapered foil (independent of C and T). The skin at the root of the wing has the same dimension as at the tip. In some cases, the skin is scaled with the dimension of the chord, gradually decreasing towards the tip. In such case, different equations have to be used (see [3] for alternative equations).

Conclusion

Brooks and Besnard provide simple and quick equations to calculate the section modulus and bending inertia of foils (both solid and hollow). These equations can be improved by a one-off numerical calculation of the bending inertia.

The method can be used for other foils or shapes like struts and beams.

For a solid ClarkY foil, the equations now become:

$$S = 0{,}0733CT^2$$

$$I = 0{,}0449CT^3$$

References

[1] E. Besnard, A. Schmitz, K. Kaups, G. Tzong, H. Hefazi, H.H. Chen, O. Kural, and T. Cebeci, "Hydrofoil Design and Optimization for Fast Ships," *Proceedings of the 1998 ASME International Congress and Exhibition*, Anaheim, CA, November 1998.

[2] Brooks, A.N., “The 20-knot Human Powered Water Craft”, Human Power, Vol.6 , No. 1, Spring 1987.

[3] Lannoy, De, S., “Section Modulus and Wing Bending Inertia of Wings”. Website: Wingbike - a Human Powered Hydrofoil.

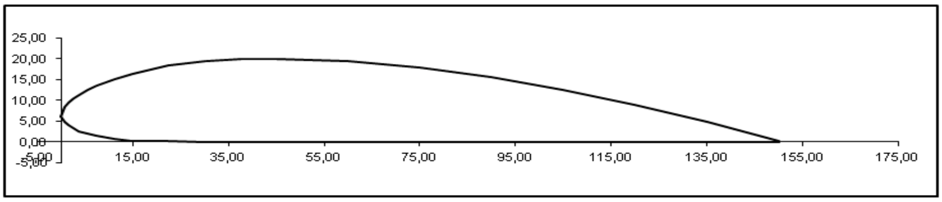
[4] Drela, M, *“Area and bending inertia of Airfoil Sections”.* MIT OpenCourseWare, Unified Engineering Course Notes, MIT Department of Aeronautics and Astronautics, http://ocw.mit.edu/courses/aeronautics-and-astronautics/16-01-unified-engineering-i-ii-iii-iv-fall-2005-spring-2006/systems-labs-06/spl10b.pdf

[5] Wikipedia: http://en.wikipedia.org/wiki/Trapezoidal_rule

Appendix 1 Numerical calculation of bending inertia

For the calculation of the section modulus and the bending inertia of foils, there are simple approximations by representing the foil cross-section by a rectangular shape. We will be following the numerical procedure as described in this paper, then we will derive all the relevant factors that lead to the rectangular cross-section approximations.

Wing Chord	150	mm
Thickness adjust +	14,0	%
Thickness =	20,00	mm



Coordinates			Trapezoidal		Neutr. S	Inertia
1	2	3	4	5	6	7
x [mm]	Zu [mm]	Zl [mm]	Area u [mm2]	Area l [mm2]	z [mm]	I [mm4]
0,00	5,98	5,98				
0,75	8,38	4,79	5,38	4,04	17,7	6,6
1,50	9,49	3,93	6,70	3,27	28,0	15,3
2,63	10,51	3,16	11,25	3,99	56,5	44,3
3,75	11,28	2,51	12,26	3,19	68,0	70,6
5,63	12,39	2,05	22,25	4,29	140,4	179,0
7,50	13,50	1,50	24,21	3,32	168,4	270,8
11,25	15,13	0,72	53,69	4,17	428,2	936,5
15,00	16,41	0,26	59,13	1,83	504,8	1337,0
22,50	18,26	0,05	130,00	1,15	1249,9	4035,3
30,00	19,42	0,00	141,28	0,19	1414,1	5128,6
37,50	19,83	0,00	147,18	0,00	1474,5	5561,5
45,00	20,00	0,00	149,36	0,00	1500,0	5750,8
60,00	19,49	0,00	296,15	0,00	2848,1	10397,3
75,00	17,98	0,00	281,03	0,00	2425,4	7676,5
90,00	15,64	0,00	252,18	0,00	1834,8	4783,8
105,00	12,56	0,00	211,54	0,00	1183,9	2892,3
120,00	8,92	0,00	161,15	0,00	597,2	2346,7
135,00	4,79	0,00	102,82	0,00	171,8	2207,0
150,00	0,21	0,00	37,44	0,00	0,3	180,6
			2105	29	16.112	53.821

area =	2.076	mm2	(sum Col. 4 - sum Col. 5)
z =	7,8	mm	(sum Col. 6 divided by area)
I =	53.821	mm4	(sum Col. 7)
S =	4.398	mm3	(I divided by y)

α =	0,294	(= correction factor for chord)
β =	1,224	(= correction factor for thickness)
S = 0,073 CT^2		(= equation for section modulus)
I = 0,045 CT^3		(= equation for bending inertia)

Appendix 2 Determining the correction factors

For the Besnard and Brooks type approximation, the chord and thickness of a foil is corrected. The chord needs to be corrected by α and the thickness is corrected with β.

The section modulus equation and bending inertia equation should be solved simultaneously to determine correction factors α and β.

$$\boxed{S = \frac{(\alpha C)(\beta T)^2}{6}} \quad \text{and} \quad \boxed{I = \frac{(\alpha C)(\beta T)^3}{12}}$$

Re-writing the first equation (S) results in: $\alpha = \dfrac{6S}{C(\beta T)^2}$

Substituting this into the next equation (I) yields: $\beta^3 = \dfrac{12I}{\alpha CT^3} = \dfrac{12I}{CT^3}\dfrac{C\beta^2 T^2}{6S}$

$$\boxed{\Rightarrow \beta = \frac{2I}{ST}}$$

Then substituting β back in the equation for α: $\alpha = \dfrac{6S}{CT^2}\left(\dfrac{ST}{2I}\right)^2 = \dfrac{6S}{CT^2}\dfrac{S^2T^2}{4I^2}$

$$\boxed{\Rightarrow \alpha = \frac{3S^3}{2CI^2}}$$

Using the values of the ClarkY foil above:

S =	4.398	mm^3
I =	53.821	mm^4
C =	150	mm
T =	20	mm

➔ α = 0,294

β = 1,224

To obtain accurate equations, it means that the chord should be corrected with 0,294 and the thickness should be corrected with 1,224. This results in the following equations for bending inertia and section modulus.

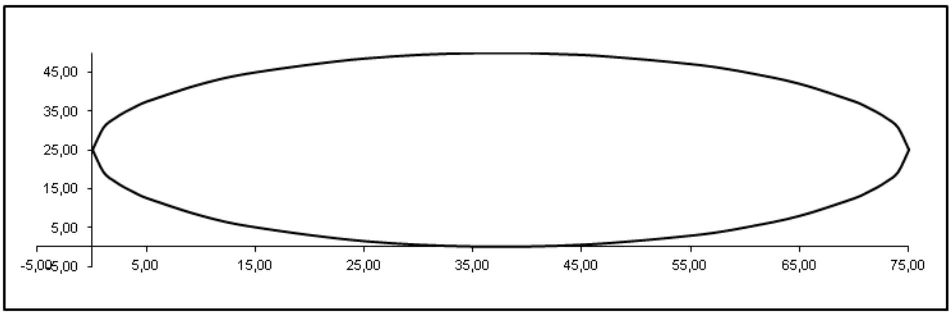
$$S = \frac{(0{,}294C)(1{,}224T)^2}{6} = 0{,}0733CT^2 \quad \text{and} \quad I = \frac{(0{,}294C)(1{,}224T)^3}{12} = 0{,}0449CT^3$$

The validity of this procedure is checked in the next appendix.

Appendix 3 Validation of the method

To illustrate the validity of the method, we have taken a foil with C=75 mm and T=50mm (actually, its an ellipse of which the area, section modulus and bending inertia can be exactly determined).

Now use the numerical method to determine the bending inertia, then solve α and β as explained in this paper.



Coordinates			Trapezoidal Rule		Neutr. S	Inertia
1	2	3	4	5	6	7
x [mm]	Zu [mm]	Zl [mm]	Area u [mm2]	Area l [mm2]	Z [mm]	I [mm4]
0,00	25,00	25,00				
1,00	30,73	19,27	27,87	22,13	286,7	125,8
2,00	33,06	16,94	31,90	18,10	402,8	348,6
3,00	34,80	15,20	33,93	16,07	489,9	627,2
4,00	36,23	13,77	35,52	14,48	561,7	945,6
5,00	37,47	12,53	36,85	13,15	623,6	1293,6
10,00	42,00	8,00	198,67	51,33	4249,2	16368,6
15,00	45,00	5,00	217,49	32,51	5000,0	26668,3
25,00	48,57	1,43	467,85	32,15	11785,1	87301,0
35,00	49,94	0,06	492,57	7,43	12472,2	103477,1
45,00	49,49	0,51	497,20	2,80	12247,4	97983,6
55,00	47,11	2,89	483,03	16,97	11055,4	72068,5
60,00	45,00	5,00	230,28	19,72	5000,0	26668,3
65,00	42,00	8,00	217,49	32,51	4249,2	16368,6
70,00	37,47	12,53	198,67	51,33	3118,0	6468,1
71,00	36,23	13,77	36,85	13,15	561,7	945,6
72,00	34,80	15,20	35,52	14,48	489,9	627,2
73,00	33,06	16,94	33,93	16,07	402,8	348,6
74,00	30,73	19,27	31,90	18,10	286,7	125,8
75,00	25,00	25,00	27,87	22,13	0,0	0,0
			3.335	415	73.282	458.760

area	2921	mm2	(sum Col. 4 - sum Col. 5)
z	25,1	mm	(sum Col. 6 divided by area)
I	458.760	mm4	(sum Col. 7)
S	18.284	mm3	(I divided by z)

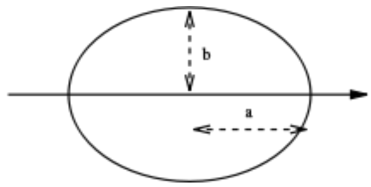
CHECK values:

Area

From Wikipedia, we can seen that the area of an ellipse is:

$$S = \frac{\pi}{2} ab$$

(S not to be confused with Section Modulus)



!!! (a=37,5 mm and b=25 mm)

The numerical scheme results in an area of 2.921mm^2, whereas the equation above results in an area of 2.945mm^2. The accuracy is 99%.

Bending inertia

From Wikipedia, we can see that the bending inertia of an ellipse is:

$$I = \frac{\pi}{4} ab^3 = 460.194mm^4$$

The numerical scheme results in a bending inertia of 458.760mm^4 (accuracy of 99%).

Section Modulus

The equation for section modulus is:

$$S = \frac{\pi}{4} ab^2 = 18.408mm^3$$

The numerical scheme results in a section modulus of 18.284mm^3 (accuracy of 99%).

Conclusion: the numerical scheme is accurate.

CHECK calculations:

Use equations 9 and 10 of this paper to determine the correction factors for chord and thickness:

$$\alpha = \frac{3S^3}{2CI^2} \quad \text{and} \quad \beta = \frac{2I}{ST}$$

With	S =	18.284	mm^3
	I =	458.760	mm^4
	C =	75	mm
	T =	50	mm

From this, it follows: α = 0,581
β = 1

So the chord has to be corrected with a factor 0,581 and the thickness has to be corrected with a factor 1 to obtain the correct equations for this ellipse. Now the equations for section modulus and bending are:

$$S = \frac{(0{,}581C)(T)^2}{6} = 0{,}0975CT^2 = 18.284mm^3$$ and

$$I = \frac{(0{,}581C)(T)^3}{12} = 0{,}0484CT^3 = 458.760mm^4$$

Conclusion: the “rectangular” equations yield the same values as the numerical scheme.

CHECK equations:

We know that a=C/2 and that b=T/2. Substituting these values into the ellipse equations:

$$I = \frac{\pi}{4}ab^3 = \frac{\pi}{4}\left(\frac{C}{2}\right)\left(\frac{T}{2}\right)^3 = \frac{\pi}{4.2.8}CT^3 = 0{,}0491CT^3 \approx 0{,}0484CT^3 = 99\%$$

$$S = \frac{\pi}{4}ab^2 = \frac{\pi}{4}\left(\frac{C}{2}\right)\left(\frac{T}{2}\right)^2 = \frac{\pi}{4.2.4}CT^2 = 0{,}0982CT^2 \approx 0{,}0975CT^2 = 99\%$$

Conclusion: the “rectangular” equations are numerically the same as the exact equations.

Appendix 3 Quick reference guide

Determine bending inertia (I) and section modulus (S) numerically

C=chord

T=thickness

$$\alpha = \frac{3S^3}{2CI^2} \qquad \beta = \frac{2I}{ST}$$

$$S = \frac{(\alpha C)(\beta T)^2}{6}$$

$$S = \frac{\alpha\beta^2}{6} CT^2$$

Section modulus equation

$$I = \frac{(\alpha C)(\beta T)^3}{12}$$

$$I = \frac{\alpha\beta^3}{12} CT^3$$

Bending inertia equation

		ClarkY	Besnard	Brooks
Section modulus	Solid foil	$S = 0{,}0733CT^2$	$S = 0{,}0903CT^2$	$S = 0{,}075CT$
	Hollow foil (uniform skin)	$S = 0{,}0733C\left\{T^2 - \frac{(T-2Sk)^3}{T}\right\}$	$S = 0{,}0903C\left\{T^2 - \frac{(T-2Sk)^3}{T}\right\}$	$S = 0{,}075C\left\{T^2 - \frac{(T-2Sk)^3}{T}\right\}$
	Hollow foil (non-uniform skin)	$S = 0{,}0733C\left\{T^2 - T^2(1-2\Omega)^3\right\}$	$S = 0{,}0903C\left\{T^2 - T^2(1-2\Omega)^3\right\}$	$S = 0{,}0705C\left\{T^2 - T^2(1-2\Omega)^3\right\}$

		ClarkY	Besnard	Brooks
Bending inertia	Solid foil	$I = 0{,}0449CT^3$	$I = 0{,}0384CT^3$	$I = 0{,}0375CT^3$
	Hollow foil (uniform skin)	$I = 0{,}0449C\left\{T^3 - (T - 2Sk)^3\right\}$	$I = 0{,}0384C\left\{T^3 - (T - 2Sk)^3\right\}$	$I = 0{,}0375C\left\{T^3 - (T - 2Sk)^3\right\}$
	Hollow foil (non-uniform skin)	$I = 0{,}0449C\left\{T^3 - T^3(1 - 2\Omega)^3\right\}$	$I = 0{,}0384C\left\{T^3 - T^3(1 - 2\Omega)^3\right\}$	$I = 0{,}0375C\left\{T^3 - T^3(1 - 2\Omega)^3\right\}$